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Conserved Charges of Higher D Kerr–AdS Spacetimes

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Abstract

We compute the energy and angular momenta of recent D-dimensional Kerr-AdS solutions to cosmological Einstein gravity, as well as of the BTZ metric, using our invariant charge definitions.

1 Introduction

Rotating solutions of cosmological Einstein gravity in D dimensions, $R_{\mu\nu} = (D-1)\,\Lambda\, g_{\mu\nu}$, have been constructed recently [1, 2], extending earlier $\Lambda = 0$ solutions of [3], themselves generalizations of the well-known D=4 metrics of [4] and [5], and of [6] in D=5. These geometries provide a useful application of our recent generalized "conserved charge" definitions, which are also extensions – of the original ADM [7], and AD [8] charges – to cover wider classes of actions [9, 11]: We will compute the energy and angular momenta of these new solutions, as well as of the $D = 3$ BTZ metric as calculated within topologically massive gravity.

Gravity theories have been historically endowed with a variety of seemingly different charge definitions, with different degrees of applicability and coordinate invariance. This topic has also seen much very recent activity, for example [12]. A summary and comparison of some of them is given in [13] which also includes a computation of the charges for Kerr-AdS black holes, using thermodynamic arguments; see also [14, 15]. Our results will agree with those, but we emphasize that in a general context, certain coincidences between charge definitions are suspect: For example, the frequently invoked "Komar" charges, are in general not applicable, being highly gauge-dependent [16].

2 Mass and Angular Momenta of Kerr-AdS

Let us briefly recapitulate the formulations of [8, 9]. The field equations of any metric model coupled to a (necessarily covariantly conserved) matter source $\tau_{\mu\nu}$ are

$$\frac{\delta I}{\delta g_{\mu\nu}} \equiv \Phi_{\mu\nu}(g, R, \nabla R, ..) = \kappa \tau_{\mu\nu}, \tag{1}$$

where $\Phi_{\mu\nu}$ is an identically conserved tensor that can depend on curvatures and their derivatives. Decompose the metric into the sum of a background "vacuum", $\bar{g}_{\mu\nu}$ (which solves (1) for $\tau_{\mu\nu} = 0$), plus a deviation $h_{\mu\nu}$, not necessarily small, that vanishes sufficiently rapidly far from the matter source: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. The field equations can be divided into a part linear in $h_{\mu\nu}$ plus a non-linear remainder, which (with $\tau_{\mu\nu}$) constitutes the total source $T_{\mu\nu}$. If the background $\bar{g}_{\mu\nu}$ admits Killing vectors $\bar{\xi}_\mu$, obeying $\bar{\nabla}_\mu\bar{\xi}_\nu + \bar{\nabla}_\nu\bar{\xi}_\mu = 0$, then, up to normalization factors (which we shall fix later), the conserved Killing charges are

$$Q^\mu(\bar{\xi}) = \int_{\mathcal{M}} d^{D-1}x\sqrt{-\bar{g}}T^{\mu\nu}\bar{\xi}_\nu = \int_\Sigma dS_i \mathcal{F}^{\mu i}. \tag{2}$$

Here Σ is a $D-2$ dimensional space-like asymptotic hypersurface of the space $\mathcal{M}$ and $\mathcal{F}^{\mu i}$ is an anti-symmetric tensor, whose explicit form is model-dependent. For Einstein's theory with a cosmological constant,

$$\begin{aligned} Q^\mu = \frac{1}{4\Omega_{D-2}G_D}\int_\Sigma \; dS_i \; \Big\{ & \bar{\xi}_\nu\bar{\nabla}^\mu h^{i\nu} - \bar{\xi}_\nu\bar{\nabla}^i h^{\mu\nu} + \bar{\xi}^\mu\bar{\nabla}^i h - \bar{\xi}^i\bar{\nabla}^\mu h \\ & + h^{\mu\nu}\bar{\nabla}^i\bar{\xi}_\nu - h^{i\nu}\bar{\nabla}^\mu\bar{\xi}_\nu + \bar{\xi}^i\bar{\nabla}_\nu h^{\mu\nu} - \bar{\xi}^\mu\bar{\nabla}_\nu h^{i\nu} + h\bar{\nabla}^\mu\bar{\xi}^i \Big\}, \end{aligned} \tag{3}$$

where i takes values in $1, 2, ...D-2$ and the charge is normalized as shown, by dividing with the D-dimensional Newton's constant and the solid angle. These charges are background gauge invariant under the diffeomorphisms $\delta_\zeta h_{\mu\nu} = \bar{\nabla}_\mu\zeta_\nu + \bar{\nabla}_\nu\zeta_\mu$: $\delta_\zeta Q^\mu = 0$.

Let us now calculate the conserved charges of the metrics [1] for $D > 3$. [We shall treat the special $D = 3$ case at the end]. They have the Kerr-Schild form [17, 18]

$$ds^2 = d\bar{s}^2 + \frac{2M}{U}\,(k_\mu\, dx^\mu)^2\,, \tag{4}$$

in terms of the de Sitter metric

$$\begin{aligned} d\bar{s}^2 \;=\; & -W\,(1-\Lambda\, r^2)\,dt^2 + F\,dr^2 + \sum_{i=1}^{N+\epsilon}\frac{r^2+a_i^2}{1+\Lambda\, a_i^2}\;d\mu_i^2 + \sum_{i=1}^{N}\frac{r^2+a_i^2}{1+\Lambda\, a_i^2}\;\mu_i^2\,d\phi_i^2 \\ & + \frac{\Lambda}{W\,(1-\Lambda\, r^2)}\Big(\sum_{i=1}^{N+\epsilon}\frac{(r^2+a_i^2)\,\mu_i\,d\mu_i}{1+\Lambda\, a_i^2}\Big)^2\,. \end{aligned} \tag{5}$$

Here $\epsilon = 0/1$ for odd/even, dimensions and $D = 2N+1+\epsilon$. The null 1-form reads

$$k_\mu\, dx^\mu \;=\; F\,dr + W\,dt - \sum_{i=1}^{N}\frac{a_i\,\mu_i^2}{1+\Lambda\, a_i^2}\,d\phi_i\,, \tag{6}$$

with

$$U \;\equiv\; r^\epsilon\sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{r^2+a_i^2}\prod_{j=1}^{N}(r^2+a_j^2),\;\; W \equiv \sum_{i=1}^{N+\epsilon}\frac{\mu_i^2}{1+\Lambda\, a_i^2},\;\; F \equiv \frac{1}{1-\Lambda\, r^2}\;\sum_{i=1}^{N+\epsilon}\frac{r^2\,\mu_i^2}{r^2+a_i^2}\,. \tag{7}$$

To find the energy and angular momenta corresponding to (4), we must compute the charges Q^0 for the corresponding Killing vectors: for the energy we shall take $\bar{\xi}^\mu = (-1, \vec{0})$ and each angular momentum has the appropriate unit entry $(0, \dots 1_i \dots 0)$. Then

$$Q^0 \;=\; \frac{1}{4\Omega_{D-2}G_D}\int_\Sigma dS_r\Big\{g_{00}\bar{\nabla}^0 h^{r0} + g_{00}\bar{\nabla}^r h^{00} + h^{0\nu}\bar{\nabla}^r\bar{\xi}_\nu - h^{r\nu}\bar{\nabla}^0\bar{\xi}_\nu + \bar{\nabla}_\nu h^{r\nu}\Big\}\,. \quad (8)$$

Using the energy Killing vector, we obtain[1]

$$\begin{aligned} E_D \;=\; \frac{1}{4\Omega_{D-2}G_D}\int_\Sigma dS_r\Big\{ & g_{00}g^{rr}\partial_r h^{00} + \frac{1}{2}h^{00}g^{rr}\partial_r g_{00} - \frac{m}{U}g^{00}\partial_r g_{00} + 2m\partial_r U^{-1} \\ & + \frac{2m}{U}g^{rr}\partial_r g_{rr} - \frac{m}{U}g^{rr}k^i k^j\partial_r g_{ij} + \frac{m}{U}g^{ij}\partial_r g_{ij}\Big\}. \end{aligned} \quad (9)$$

To compute E_D, one needs the large r behavior of the integrand I of (9); since

$$g_{00} \to W\Lambda r^2, \qquad F \to \frac{-1}{\Lambda r^2}, \qquad U \to r^{D-3}, \qquad k^\phi \;\to \tfrac{a_\phi}{r^2}\;, \quad (10)$$

then

$$I = \frac{2m}{r^{D-2}}[(D-1)W - 1]\;. \quad (11)$$

For completeness, let us also note how the determinant is calculated,

$$\det g = -W(1-\Lambda r^2)F\prod_{i=1}^{N}\frac{(r^2+a_i^2)\mu_i^2}{1+\Lambda a_i^2}\,\det M\;. \quad (12)$$

Here M is the matrix representing the coefficients of the form $d\mu_i d\mu_j$ in the metric, which can be expressed as (no repeated index summation),

$$M_{ij} = A_i\delta_{ij} + B_iB_j + C_iC_j \quad (13)$$

where

$$\begin{aligned} A_i \;&=\; \frac{(r^2+a_i^2)}{1+\Lambda a_i^2}, \qquad\qquad B_i = \sqrt{\frac{(r^2+a_{N+\epsilon}^2)}{1+\Lambda a_{N+\epsilon}^2}}\,\frac{\mu_i}{\mu_n} \\ C_i \;&=\; \sqrt{\frac{\Lambda}{W(1-\Lambda r^2)}}\Big(\frac{(r^2+a_i^2)}{1+\Lambda a_i^2} - \frac{(r^2+a_{N+\epsilon}^2)}{1+\Lambda a_{N+\epsilon}^2}\Big)\mu_i\;. \end{aligned} \quad (14)$$

[1]We are assuming that the background spacetime is AdS rather than dS, whose cosmological horizon causes complications. Some of these issues were addressed in [8, 9]. For details of acceptable asymptotic falloff to (A)dS in various dimensions, we refer to [10].

Then we have

$$\det M = \prod_{i=1}^{N+\epsilon-1} A_i \sum_{i=1}^{N+\epsilon-1} \left\{ \frac{B_i^2}{A_i} + \frac{C_i^2}{A_i} + \sum_{j\neq i}^{N+\epsilon-1} \frac{B_i^2 C_i^2}{A_i A_j} - \sum_{j\neq i}^{N+\epsilon-1} \frac{B_i B_j C_j C_i}{A_i A_j} \right\} . \tag{15}$$

Inserting (14) in the above equation, one gets

$$\det M = \frac{1}{W \mu_{N+\epsilon}^2} \prod_{i=1}^{N} \frac{1}{1+\Lambda a_i^2} \ . \tag{16}$$

Using equations (16,12,11) the energy of the D dimensional rotating black hole becomes

$$E_D = \frac{m}{\Xi} \sum_{i=1}^{\frac{D-1-\epsilon}{2}} \left\{ \frac{1}{\Xi_i} - (1-\epsilon)(\frac{1}{2}) \right\} . \tag{17}$$

where

$$\Xi \equiv \prod_{i=1}^{\frac{D-1-\epsilon}{2}} (1+\Lambda a_i^2), \quad \Xi_i \equiv 1 + \Lambda a_i^2. \tag{18}$$

This expression reduces to the standard limits $a_i \to 0$ and $\Lambda \to 0$, and agrees (up to a constant factor) with those of [13, 14].

The computation of angular momenta follows along similar lines. Consider a given, say that i^{th} (which we call the ϕ) component, *i.e.*, the Killing vector $\xi^\mu_{(i)} = (0, ..., 0, 1_i, 0, ..)$. Then the corresponding Killing charge becomes

$$\begin{aligned} Q^0 &= \frac{1}{4\Omega_{D-2} G_D} \int_\Sigma dS_r \Big\{ g_{\phi\phi} \bar{\nabla}^0 h^{r\phi} - g_{\phi\phi} \bar{\nabla}^r h^{0\phi} + h^{0\nu} \bar{\nabla}^r \bar{\xi}_\nu - h^{r\nu} \bar{\nabla}^0 \bar{\xi}_\nu \Big\} \\ &= \frac{1}{4\Omega_{D-2} G_D} \int_\Sigma dS_r \Big\{ - g_{\phi\phi} g^{rr} g^{00} \partial_r h_0^{\ \phi} \Big\} \ . \end{aligned} \tag{19}$$

Once again the integrand can be calculated to be

$$I = \frac{(D-1) 2 m a_i \mu_i^2}{r^{D-2}(1+\Lambda a_i^2)}. \tag{20}$$

Putting the pieces together, the angular momentum is

$$J_i = \frac{m a_i}{\Xi \Xi_i}. \tag{21}$$

This expression again agrees with [13, 14]. Note that, unlike in the energy expression, ϵ does not appear here since even dimensional spaces have as many independent 2-planes as the odd dimensional spaces with one lower dimension.[2]

Having computed the desired conserved charges (17,21) for Kerr-AdS spacetimes in $D > 3$, let us briefly turn our attention to the $D = 3$ BTZ black hole [19]. This solution has long been studied but we recompute the charges with our method for the sake of completeness. The BTZ black hole differs from its higher dimensional counterparts in one very important aspect: for it, AdS is not the correct-vacuum-background [19]. The full metric is

$$ds^2 = (M - \Lambda r^2)dt^2 + \frac{dr^2}{-M + \Lambda r^2 + \frac{a^2}{4r^2}} - a dt\, d\phi + r^2 d\phi^2 \ , \tag{22}$$

The background metric corresponds to $M = 0$ and AdS corresponds to $M = -1$. Only AdS with $J = 0$ is allowed for $M < 0$: the others have naked singularities. So we consider $M > 0$ and compute the charges following our calculations above (about the $M = 0$ background.) We get the usual answers

$$E = M \ , \qquad\qquad J = a \ . \tag{23}$$

BTZ black holes also solve the more general topologically massive gravity equations, where the Einstein term is augmented by the Cotton tensor [20],

$$G_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = \kappa \tau_{\mu\nu} \ . \tag{24}$$

Conserved charges in this model were obtained in [21], in terms of those of the Einstein model Q^μ_E,

$$\begin{aligned} Q^\mu(\bar{\xi}) \ &= \ Q^\mu_E(\bar{\xi}) + \frac{1}{2\mu} \oint dS_i \left\{ \epsilon^{\mu i \beta} \mathcal{G}^L{}_{\nu\beta} \bar{\xi}^\nu + \epsilon^{\nu i}{}_\beta \mathcal{G}_L{}^{\mu\beta} \bar{\xi}_\nu + \epsilon^{\mu\nu\beta} \mathcal{G}^{L\,i}{}_\beta \bar{\xi}_\nu \right\} \\ &\quad + \frac{1}{2\mu} Q^\mu_E(\epsilon \bar{\nabla} \bar{\xi}) \ , \end{aligned} \tag{25}$$

[2] For even dimensions, there is a nice relation between the energy and the angular momentum $E = \sum_i \frac{J_i}{a_i}$.

where $Q^\mu_E(\epsilon\bar{\nabla}\bar{\xi})$ is the Einstein form but $\bar{\xi}$ is replaced with its curl. Once the contributions of the Cotton parts are computed the mass and the angular momentum of the BTZ black hole reads:

$$E = M - \frac{\Lambda a}{\mu}, \qquad J = a - \frac{M}{\mu}, \tag{26}$$

a shift in values that may be compared with those for gravitational anyons [22], (linearized) solutions of TMG but not of pure D=3 Einstein.

3 Mass and Angular Momenta in Higher Curvature Models

We turn now to a slightly more formal exercise, which is to indicate the stability of our generic charge definition framework as it applies to a wider range of models, specifically higher derivative gravities. While Kerr-like solutions to $R+R^2$ gravity models have yet to be discovered, it is not unlikely that they would approach the Einstein ones asymptotically. In that case, we could compute their conserved charges-defined as integrals at infinity, using the definitions for generic quadratic models [9]. Let us stick to the quadratic models of the form[3]

$$I = \int d^D x\sqrt{-g}\Big\{\frac{R}{2\kappa} + 2\Lambda_0 + \alpha R^2 + \beta R^2_{\mu\nu} + \gamma(R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2)\Big\}\ . \tag{27}$$

This model allows constant curvature spacetimes with an effective cosmological constant given as

$$\Lambda = -\frac{1}{4f(\alpha,\beta,\gamma_)\kappa}\left\{1 \pm \sqrt{1+8\kappa f(\alpha,\beta,\gamma)\Lambda_0}\right\} \quad \text{for} \quad f(\alpha,\beta,\gamma) \neq 0\ , \tag{28}$$

where

$$f(\alpha,\beta,\gamma) = \frac{(D-4)}{(D-2)^2}(D\alpha+\beta) + \frac{\gamma(D-4)(D-3)}{(D-2)(D-1)}\ . \tag{29}$$

[3]Note that we changed normalization of the cosmological constant compared to the previous section.

When the bare cosmological constant vanishes ($\Lambda_0 = 0$), (A)dS spaces are still allowed and one has the + sign branch in (28). Conserved charges in this model, which we quote below, were defined in [9]

$$\begin{aligned}
Q^\mu(\bar\xi) \;&=\; \left\{\frac{1}{\kappa}+\frac{4\Lambda D\alpha}{D-2}+\frac{4\Lambda\beta}{D-1}+\frac{4\Lambda\gamma(D-4)(D-3)}{(D-2)(D-1)}\right\}\int d^{D-1}\,x\sqrt{-\bar g}\bar\xi_\nu\mathcal{G}_L^{\mu\nu} \\
&+(2\alpha+\beta)\int dS_i\sqrt{-g}\Big\{\bar\xi^\mu\bar\nabla^i R_L + R_L\bar\nabla^\mu\,\bar\xi^i - \bar\xi^i\bar\nabla^\mu R_L\Big\} \\
&+\beta\int dS_i\sqrt{-g}\Big\{\bar\xi_\nu\bar\nabla^i\mathcal{G}_L^{\mu\nu} - \bar\xi_\nu\bar\nabla^\mu\mathcal{G}_L^{i\nu} - \mathcal{G}_L^{\mu\nu}\bar\nabla^i\bar\xi_\nu + \mathcal{G}_L^{i\nu}\bar\nabla^\mu\bar\xi_\nu\Big\}\,. \qquad (30)
\end{aligned}$$

where $\mathcal{G}_L^{\mu\nu}$ and R_L are the linear parts of the Einstein tensor and the scalar curvature, respectively. The second and the third line vanish for Einstein spaces. The first line, on the other hand is just a factor times the cosmological Einstein theory's charges (3). Therefore for asymptotic Kerr-AdS solutions, their conserved charges are given by the first term in (30), under the condition (28). Let us specifically consider the popular Einstein–Gauss–Bonnet theory, $\alpha = \beta = 0$. Also, implementing the condition (28) (with the + sign) we have,

$$Q^\mu = -\sqrt{1+8\kappa f(\gamma,0,0)\Lambda_0}\frac{1}{\kappa}\int d^{D-1}\,x\sqrt{-\bar g}\bar\xi_\nu\mathcal{G}_L^{\mu\nu}\;. \qquad (31)$$

Although the energy seems to have the wrong sign, this is a red herring: As shown in [23], for the non-rotating case the exact metric reads

$$ds^2 = g_{00}dt^2 + g_{rr}dr^2 + r^2 d\Omega_{D-2} \qquad (32)$$

$$-g_{00} = g_{rr}^{-1} = 1 + \frac{r^2}{4\kappa\gamma(D-3)(D-4)}\left\{1 \pm \left\{1 + 32\gamma\kappa(D-3)(D-4)\frac{m}{(D-2)r^{D-1}}\right\}^{\frac{1}{2}}\right\}, \qquad (33)$$

whose asymptotic forms branch into Schwarzschild and Schwarzschild-de-Sitter respectively,

$$-g_{00} = 1 - \frac{r_0}{r^{D-3}}, \qquad -g_{00} = 1 + \frac{4m}{(D-2)}r^{D-3} + \frac{r^2}{\gamma(D-3)(D-4)}. \qquad (34)$$

We see that SdS branch comes with the “wrong” sign compared to the usual Schwarzschild one. Therefore, the minus sign in the energy becomes positive once $\mathcal{G}_L^{\mu\nu}$ is explicitly computed. We conclude that the conserved charges in the Einstein–Gauss–Bonnet theory for such asymptotic solutions would be simply proportional to those of (17, 21) cosmological gravity:

$$E_{\rm GB} = \sqrt{1+8\kappa f(\gamma,0,0)\Lambda_0}\, E_D \qquad\qquad J_i({\rm GB}) = \sqrt{1+8\kappa f(\gamma,0,0)\Lambda_0}\, J_i, \qquad (35)$$

It is important to note that if the coefficient $\sqrt{1+8\kappa f(\gamma,0,0)\Lambda_0}$ does not vanish, then one can simply rescale the Killing charges to get the Einstein charges (17, 21).

4 Conclusions

Using the charge definitions via background Killing charges of [8, 9] we have computed the mass and angular momenta of the rotating Kerr-AdS black holes for D dimensions for cosmological Einstein gravity. As a test of stability, we checked that the corresponding charge definitions for higher order would lead to the same values for asymptotically similar geometries up to the indicated constant rescaling.

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